Conclusion

Closed-form solutions for finding the largest possible hypersphere of stability for third- and fourth-order Hurwitz polynomials are reported. These formulas are very convenient to use and reduce computational effort to a minimum. They can be employed to find bounds under weakly structured perturbations as well as highly structured perturbations.

Appendix

The three sets of simultaneous polynomials in Table 2 are reduced to single polynomials that are shown here. Since a special form of the Euclidean algorithm[12] is utilized, the resultant polynomials have more roots than the original two polynomials. These extra roots can be easily identified and eliminated by back substitution in the original polynomials. These equations were derived with the help of *Mathematica*[11] and verified numerically against their corresponding equations in Table 2.

Equation (*) can be solved by solving:

$$(\gamma \boldsymbol{b}_3 - 2\beta)(\gamma \boldsymbol{b}_3 - \beta)[\boldsymbol{b}_3^3 + 2(\alpha - \gamma^2 - 1)\boldsymbol{b}_3 + 2(\delta + \beta\gamma)][\boldsymbol{b}_3^7 - \delta \boldsymbol{b}_3^6 + 2(2\alpha + \gamma^2 - 2)\boldsymbol{b}_3^5 + (8\delta - 4\alpha\delta - 9\beta\gamma + \alpha\beta\gamma - 2\delta\gamma^2)\boldsymbol{b}_3^4 + (4 - 8\alpha + 4\alpha^2 + 8\beta^2 - 2\alpha\beta^2 - 4\delta^2 + 9\beta\gamma\delta + 5\gamma^2 - 2\alpha\gamma^2 - \beta^2\gamma^2 + \alpha^2\gamma^2 + \gamma^4)\boldsymbol{b}_3^3 + (16\alpha\delta - 12\delta - 4\alpha^2\delta - 8\beta^2\delta - 6\beta\gamma - 2\alpha^2\beta\gamma - 3\beta^3\gamma - 10\delta\gamma^2 + 2\alpha\delta\gamma^2 - 3\beta\gamma^3 - \alpha\beta\gamma^3 - \delta\gamma^4)\boldsymbol{b}_3^2 + (4\alpha\beta^2 - 2\beta^4 + 12\delta^2 + 12\delta^2 - 8\alpha\delta^2 + 2\beta^2\gamma^2 + 3\alpha\beta^2\gamma^2 + 5\delta^2\gamma^2 - 3\beta\delta\gamma^3 + 12\beta\gamma\delta)\boldsymbol{b}_3 - (2\alpha\beta^3\gamma + 4\alpha\beta^2\delta + 2\beta^2\delta\gamma^2 + 6\beta\gamma\delta^2 + 4\delta^3)] = 0$$

Equation (**) can be solved by solving:

$$\boldsymbol{b}_3^8 - 4\delta \boldsymbol{b}_3^7 + 2(3\delta^2 - \alpha^2 + 1)\boldsymbol{b}_3^6 + (\beta\gamma - \alpha\beta\gamma - 8\delta - 4\delta^3 + 4\alpha^2\delta)\boldsymbol{b}_3^5 + (1 - 2\alpha^2 + \alpha^4 - \beta^2 + \alpha^2\beta^2 + \gamma^2 + 2\alpha\gamma^2 + \alpha^2\gamma^2 - 3\beta\gamma\delta + 2\alpha\beta\gamma\delta + 12\delta^2 - 2\alpha^2\delta^2 + \delta^4)\boldsymbol{b}_3^4 + (-4\delta - \beta\gamma - 3\alpha\beta\gamma - 3\alpha^2\beta\gamma - \alpha^3\beta\gamma + 4\alpha^2\delta + 3\beta^2\delta - \alpha^2\beta^2\delta - 3\gamma^2\delta - 4\alpha\gamma^2\delta - \alpha^2\gamma^2\delta + 3\beta\gamma\delta^2 - \alpha\beta\gamma\delta^2 - 8\delta^3)\boldsymbol{b}_3^3 + (4\alpha^2\beta^2 - \alpha\beta^2\gamma^2 + 3\beta\gamma\delta + 6\alpha\beta\gamma\delta + 3\alpha^2\beta\gamma\delta + 6\delta^2 - 2\alpha^2\delta^2 - 3\beta^2\delta^2 + 3\gamma^2\delta^2 + 2\alpha\gamma^2\delta^2 - \beta\gamma\delta^3 + 2\delta^4)\boldsymbol{b}_3^2 + (\alpha\beta^3\gamma + \alpha^2\beta^3\gamma - 4\alpha^2\beta^2\delta + \alpha\beta^2\gamma^2\delta - 3\beta\gamma\delta^2 - 3\alpha\beta\gamma\delta^2 - 4\delta^3 + \beta^2\delta^3 - \gamma^2\delta^3)\boldsymbol{b}_3 + (\beta\gamma\delta^3 + \delta^4 - \alpha\beta^3\gamma\delta - \alpha^2\beta^4) = 0$$

Equation (***) can be solved by solving:

$$[\beta \boldsymbol{b}_3^4 + (2\gamma - \alpha\gamma - \beta\delta)\boldsymbol{b}_3^3 + \gamma\delta \boldsymbol{b}_3^2 + \beta(\beta\gamma - 2\gamma)\boldsymbol{b}_3 - \beta^3][A\boldsymbol{b}_3^5 - \delta A\boldsymbol{b}_3^4 + (2\beta^2 - 4\alpha\beta^2 + 2\alpha^2\beta^2 + 2\beta^4 + \beta^2\gamma^2 - 2\alpha\beta^2\gamma^2 - \alpha^2\beta^2\gamma^2 - 5\beta\gamma\delta + 3\alpha\beta\gamma\delta + \alpha^2\beta\gamma\delta + \alpha^3\beta\gamma\delta + \beta^3\gamma\delta - \alpha\beta^3\gamma\delta - 3\beta\gamma^3\delta - \alpha\beta\gamma^3\delta - 2\delta^2 + 4\alpha\delta^2 - 2\alpha^2\delta^2 + 5\beta^2\delta^2 - 2\alpha\beta^2\delta^2 + \alpha^2\beta^2\delta^2 - \gamma^2\delta^2 + 2\alpha\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 - \beta^2\gamma^2\delta^2 - \beta\gamma\delta^3 + \alpha\beta\gamma\delta^3 + 2\delta^4)\boldsymbol{b}_3^3 + (-3\beta^3\gamma + 2\alpha\beta^3\gamma + \alpha^2\beta^3\gamma + \beta^5\gamma - \beta^3\gamma^3 - 6\beta^2\delta + 12\alpha\beta^2\delta - 6\alpha^2\beta^2\delta - 2\beta^4\delta + 6\alpha\beta^2\gamma^2\delta + 9\beta\gamma\delta^2 - 6\alpha\beta\gamma\delta^2 - 3\alpha^2\beta\gamma\delta^2 + 2\beta^3\gamma\delta^2 + 3\beta\gamma^3\delta^2 + 2\delta^3 - 4\alpha\delta^3 + 2\alpha^2\gamma^3 - 4\beta^2\delta^3 - 2\alpha\gamma^2\delta^3 + \beta\gamma\delta^4 - 2\delta^5)\boldsymbol{b}_3^2 + (2\alpha\beta^4 - \alpha^2\beta^4 - \beta^6 + \beta^4\delta^2 + \alpha\beta^4\delta^2 + 6\beta^3\gamma\delta - 5\alpha\beta^3\gamma\delta + \alpha^2\beta^3\gamma\delta + \beta^3\gamma^3\delta + 6\beta^2\delta^2 - 8\alpha\beta^2\delta^2 + 4\alpha^2\beta^2\delta^2 - 2\beta^4\delta^2 - \beta^2\gamma^2\delta^2 - \alpha\beta^2\gamma^2\delta^2 - 3\beta\gamma\delta^3 + 3\alpha\beta\gamma\delta^3 + \delta^4 - \beta^2\delta^4 + \gamma^2\delta^4)\boldsymbol{b}_3 + (-\alpha\beta^5\gamma - 2\alpha\beta^4\delta + \alpha^2\beta^4\delta - \beta^4\gamma^2\delta - 3\beta^3\gamma\delta^2 + \alpha\beta^3\gamma\delta^2 - 2\beta^2\delta^3 - \beta\gamma\delta^4 - \delta^5)] = 0;\ A = (1 - 4\alpha + 6\alpha^2 - 4\alpha^3 + \alpha^4 + 2\beta^2 - 4\alpha\beta^2 + 2\alpha^2\beta^2 + \beta^4 + 2\gamma^2 - 4\alpha\gamma^2 + 2\alpha^2\gamma^2 - 2\beta^2\gamma^2 + \gamma^4 - 8\beta\gamma\delta + 8\alpha\beta\gamma\delta - 2\delta^2 + 4\alpha\delta^2 - 2\alpha^2\delta^2 + 2\beta^2\delta^2 + 2\gamma^2\delta^2 + \delta^4)$$

References

[1]Yeung, K. S., and Chaid, K. Q., "Determination of Largest Hypersphere of Stability Using Lagrange Multipliers," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 3, 1992, pp. 791-794.

[2]Soh, C. B., Berger, C. S., and Dabke, K. P., "On the Stability Properties of Polynomials with Perturbed Coefficients," *IEEE Transactions on Automatic Control,* Vol. AC-30, No. 10, 1985, pp. 1033-1036.

[3]Kharitonov, V. L., "Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations," *Differential Equations,* Vol. 14, No. 11, 1978, pp. 1483-1485.

[4]Anderson, B. D. D., Jury, E. I., and Mansour, M., "On Robust Hurwitz Polynomials," *IEEE Transactions on Automatic Control,* Vol. AC-32, No. 10, 1987, pp. 909-913.

[5]Patel, R. V., and Toda, M., "Quantitative Measure of Robustness for Multivariable Systems," *Proceedings of the Joint Automatic Control Conference* (San Francisco, CA), 1980, pp. TP8-A.

[6]Yedavalli, R. K., "Improved Measures of Stability Robustness for Linear State Space Models," *IEEE Transactions on Automatic Control,* Vol. AC-30, No. 6, 1985, pp. 577-579.

[7]Yedavalli, R. K., "Perturbation Bounds for Robust Stability in Linear State Space Models," *International Journal of Control,* Vol. 42, No. 6, 1985, pp. 1507-1517.

[8]Juang, Y., Kuo, T., and Hsu, C., "Stability Robustness Analysis for State Space Models," *Proceedings of 25th Conference on Decision and Control* (Athens, Greece), Dec. 1986, pp. 745-750.

[9]Jury, E. I., and Pavilidis, T., "Stability and Aperiodicity Constraints for System Design," *IEEE Transactions on Circuit Theory,* March 1963, pp. 137-141.

[10]Horisberger, H. P., and Belanger, P. R., "Regulators for Linear Time Invariant Plants with Uncertain Parameters," *IEEE Transactions on Automatic Control,* Oct. 1976, pp. 705-708.

[11]Wolfram, S., *Mathematica: A System for Doing Mathematics by Computer,* Addison-Wesley, Reading, MA, 1988.

[12]Bocher, M., *Introduction to Higher Algebra,* Dover, New York, 1964.

[13]Nelson, R. C., *Flight Stability and Automatic Control,* McGraw-Hill, New York, 1989.

New Analytical Solutions for Proportional Navigation

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I. Introduction

PROPORTIONAL navigation (PN) is accepted as a celebrated guidance law for many guided missile applications like the surface-to-air, air-to-air, and air-to-surface missile encounters, standoff weapon delivery, and space rendezvous. In earlier studies, the guided point (pursuer) is assumed to move toward a target point in a plane containing the velocity vectors of the two points.[1] The PN strategy is defined such that the velocity vector (heading) of the pursuer is turned at a rate proportional to the rotation rate of the the line joining the pursuer and target which is the line of sight (LOS). The PN principle helps to estimate the magnitude of the lateral acceleration as a function of LOS turn rate. Three different types of PN have been defined: pure proportional navigation (PPN), true proportional navigation (TPN), and generalized true proportional navigation (GTPN) (see Fig. 1).[7]

All cases of PN are described by highly nonlinear equations of motion. Currently, there is a great deal of interest in obtain-

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ing closed-form solutions for the nonlinear equations of motion governing the PN law.[3] Parallel to this, optimal guidance schemes have also been studied in great deal. Optimal guidance leads to a nonlinear two-point boundary-value problem whose solution has been obtained analytically in closed form by Lin and Tsai,[4] Rao,[5] and Hough.[6]

Yang et al.[8] have extended the results of TPN to any angle between the missile acceleration and the LOS GTPN. Their GTPN closed-form results are restricted to cases where the theoretical acceleration gain constant K_r is zero. This Note presents the closed-form solution of TPN where $K_r \neq 0$. This derivation is presented in Sec. II. The generalization presented for GTPN here will be able to handle any function of line-of-sight angle.

We hope the results will provide new insights into the existing analytical solution on proportional navigation.

II. Analytical Solution of GTPN for $K_r \neq 0$

Recently, Yang and Yeh[8] have formulated a general proportional navigation law for homing missiles. This law is in a generalized form; the other existing proportional navigation laws are derived as particular cases, namely, the TPN law. Yang and Yeh show that the GTPN law is considered more advantageous than the existing TPN law and also leads to greater capture area and shorter intercept time.

The relevant equations are expressed as

$$\frac{\mathrm{d}V_r}{\mathrm{d}\theta}, \qquad \frac{\mathrm{d}V_\theta}{\mathrm{d}\theta}$$

when compared to the formulation in terms of usual acceleration, namely,

$$\left[\ddot{r} - r\dot{\theta}^2, \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\right]$$

This alternative approach as formulated by Yang and Yeh leads to the closed-form solution; the TPN case as formulated by Guelman[2] forms the particular case of their study, which is formulated in terms of acceleration along and normal to the radius vector.

Yang and Yeh claim that generalized proportional navigation, when formulated in terms of acceleration cannot be solved analytically in the presence of $K_r \neq 0$ (Ref. 8).

The present Note shows the existence of the closed-form solution when the governing equations are expressed in terms of the acceleration.

From both formulations, TPN has been solved analytically in terms of position as an instantaneous function of time to intercept.

In this study, all of the notations and assumptions are the same as in Ref. 8; new terms will be defined at the appropriate places.

Method of Solution

The differential equations (4a), (4b), (5a), and (5b) of Ref. 8 are taken for the study and analyzed for results. They are given as follows:

System I:

$$\frac{\mathrm{d}V_r}{\mathrm{d}t} = (V_\theta - K_r)\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{1a}$$

$$\frac{\mathrm{d}V_0}{\mathrm{d}t} = (-V_r - K_\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{1b}$$

System II:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = -K_r\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{2a}$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = -K_\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{2b}$$

$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

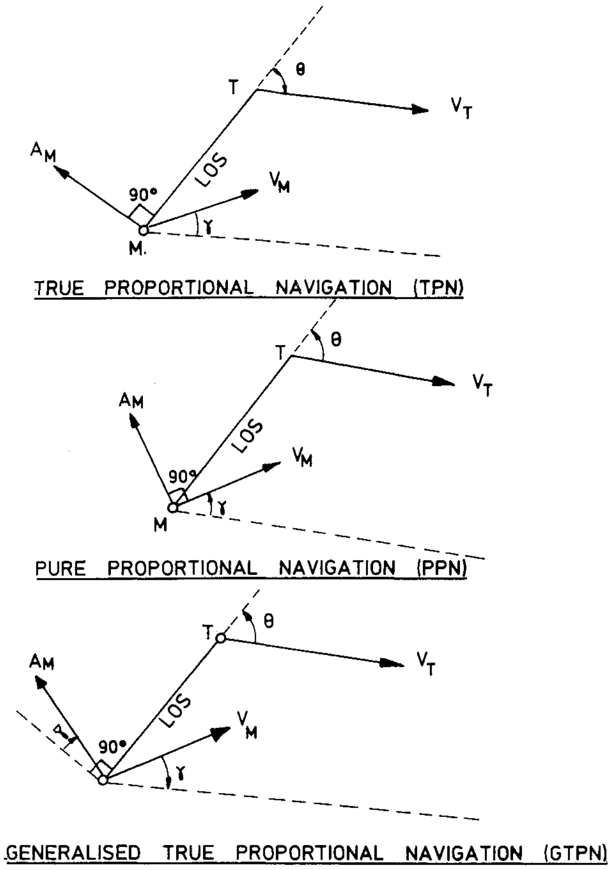

Fig. 1 Types of proportional navigation.

System I or II is the system governing the generalized proportional navigation. System I can be solved easily once the independent variable t is changed to θ.

Solution of System I

System I becomes

$$\frac{\mathrm{d}V_r}{\mathrm{d}\theta} = (V_\theta - K_r) \tag{3}$$

$$\frac{\mathrm{d}V_\theta}{\mathrm{d}\theta} = (-V_r - K_\theta) \tag{4}$$

Equations (3) and (4) are the same as Eqs. (6a) and (6b) of Ref. 8.

From Eqs. (3) and (4), we get

$$\frac{\mathrm{d}V_r}{\mathrm{d}V_\theta} = \frac{V_\theta - K_r}{-V_r - K_\theta} \tag{5}$$

The differential equation (5) is easily solved as the variables are separable. Hence,

$$-(V_r^2/2) - K_\theta V_r = (V_\theta^2/2) - K_\theta V_\theta + C \tag{6}$$

The constant of integration C can be easily determined with the aid of the condition as taken from Ref. 8. Hence,

$$(V_r + K_\theta)^2 + (V_\theta - K_r)^2 = R^2 \tag{7}$$

where

$$R^2 = (V_{r_0} + K_\theta)^2 + (V_{\theta_0} - K_r)^2$$

Hence, Eq. (7) describes the circle with center $(-K_\theta, +K_r)$ and radius R. In Ref. 8, the relation between V_θ and V_r is expressed in parametric form, namely,

$$v = a \cos\phi + \lambda$$

cf. Eqs. (8a) and (8b) of Ref. 8

$$v = b \sin\phi + \lambda_2$$

This is obtained after nondimensionalizing V_θ and V_r.

Thus, the further relation between angular momentum and t and angular momentum and the aspect angle θ is derived. It is shown here that it is possible to obtain the closed-form solution with system II when $K_r \neq 0$.

Solution of System II

$$\frac{\mathrm{d}r^2}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = -K_r\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{8a}$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot\theta) = -K_\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{8b}$$

By decoupling Eqs. (8a) and (8b), we get

$$(r'+K_\theta)\left[a+\sqrt{a^2+4rr''}\right] + 2r\left[\frac{r'r''+rr'''}{\sqrt{a^2+4rr''}}\right] = 0 \tag{9}$$

where

$$a = K_r, \qquad r' = \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$r'' = \frac{\mathrm{d}_r^2}{\mathrm{d}\epsilon^2}, \qquad r''' = \frac{\mathrm{d}_r^3}{\mathrm{d}t^3}$$

The differential equation (9) is the third-order (nonlinear) equation in r, and K_θ and K_r are as defined in Ref. 8.

The solution of Eq. (9) must be obtained. This will establish that system II is solvable in closed-form when $K_r \neq 0$.

It can be seen that the relation

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + 2K_\theta\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) + (K_\theta)^2 + \left(\frac{a+\sqrt{a^2+4rr''}}{2}\right)^2 - K_r\left(a+\sqrt{a^2+4rr''}\right) + (K_r)^2 = R^2 \tag{10}$$

is the first integral of Eq. (9) that is also the solution of the system of Eqs. (3) and (4). It can also be easily verified that the derivative of the relation (10) leads to Eq. (9) after some manipulation, thus establishing the relation (10) as the first integral of Eq. (9).

The case of TPN that corresponds $K_r = 0$ is obtained from system II. When $K_r = a = 0$, the differential equation (9) becomes

$$(r'+K_\theta)\left(2\sqrt{rr''}\right) + 2r\left[\frac{r'r''+rr'''}{2\sqrt{rr''}}\right] = 0 \tag{11}$$

After simplification, it further reduces to

$$3r'r'' + rr''' + 2K_\theta r'' = 0 \tag{12}$$

Alternatively,

$$3r'r'' + rr''' + 2K_\theta r'' = 0$$

Equation (12) is a nonlinear equation of third order. This can be solved in closed-form giving r as a function of t.

The solution of Eq. (12) is

$$\frac{(M_1)^{\mu-1/2}}{(M_2)^{\mu+1/2}} = KT^{\lambda_1^2} \tag{13}$$

where

$$M_1 = \left(V+K_\theta-\sqrt{1+K_\theta^2}\right), \qquad M_2 = \left(V+K_\theta+\sqrt{1+K_\theta^2}\right)$$

$$V = r/T; \qquad T = \lambda_1^2 t + \lambda_2$$

$$\mu = K_\theta/2\sqrt{1+K_\theta^2}$$

λ_1^2, λ_2, and K are constants. λ_1 and λ_2 are different from those defined in Ref. 8.

When $K_r = 0$, we get from Eq. (10)

$$r'^2 + K_\theta^2 + 2K_\theta r' + rr'' = R^2 \tag{14}$$

which also admits the solution as given by Eq. (13).

Thus, it is seen that when $K_r = 0$, i.e., the TPN case, the position can be expressed as an instantaneous function of time.

Analytical Solution of System II when $K_r \neq 0$

The decoupled equation (9) admits the first integral given by Eq. (10), namely,

$$r'^2 + 2K_\theta r' + (K_\theta)^2 + \left(\frac{a+\sqrt{a^2+4rr''}}{2}\right)^2 - K_r\left(a+\sqrt{a^2+4rr''}\right) + (K_r)^2 = R^2 \tag{15}$$

This can be written also as

$$(r\dot\theta) = a + \sqrt{R^2-(r'+K_\theta)^2} \tag{16}$$

Solving Eq. (8a) for $(r\dot\theta)$ and from Eq. (16), we get

$$rr'' = a\sqrt{R^2-(r'+K_\theta)^2} + \left[R^2-(r'+K\theta)^2\right] \tag{17}$$

The differential equation can be solved relating r and $\mathrm{d}r/\mathrm{d}t$. By changing the independent variable t to r and writing

$$\frac{\mathrm{d}r}{\mathrm{d}t} = P$$

and letting

$$(P+K_\theta) = R\sin\theta$$

we get the solution of the differential equation as

$$\frac{\left[\sqrt{(R^2-K_r^2)}\tan\theta/2-(R+K_r)\right]^\mu}{(R\cos\theta+K_r)\left[\sqrt{(R^2-K_r^2)}\tan\theta/2-(R+K_r)\right]^\mu} = K_1 r \tag{18}$$

where

$$\mu = K_\theta/\sqrt{R^2-K_r^2}, \qquad \lambda = (P+K_\theta)/R$$

$$\tan\theta/2 = \frac{1+\sqrt{1-\lambda^2}}{\lambda}, \qquad K_1 = \mathrm{const}$$

Equation (18) relates the position in terms of its rate, i.e., $\mathrm{d}r/\mathrm{d}t$, whereas the relation of Ref. 8 relates angular momentum in terms of position and time.

Thus, it is clear that the generalized proportional navigation when described by system II, namely, by Eqs. (8a) and (8b), is also solvable in closed form when $K_r \neq 0$.

III. More Generalized Closed-Form Solutions of GTPN

The equations of motion governing the proportional navigational course as given by Eq. (6a) and (6b) of Ref. 8 are

$$\dot{V}_r - V_\theta \dot{\theta} = F(\theta)\dot{\theta}$$

$$\dot{V}_\theta + V_r \dot{\theta} = G(\theta)\dot{\theta}$$

These are the nonlinear differential equations whose solution must be obtained to get the trajectory history of the missile following this type of law.

In Ref. 8, a closed-form solution to the preceding equations is presented by taking the functions $F(\theta)$ and $G(\theta)$ as

$$F(\theta) = -K\left[f'(\theta) - g(\theta)\right] \tag{19}$$

$$G(\theta) = -K\left[f(\theta) + g'(\theta)\right] \tag{20}$$

where K is the proportionality constant and prime denotes the differentiation with respect to θ.

By changing the independent variable from t to θ, the system (6a) and (6b) takes the form,

$$V_r' - V_\theta = -K\left[f'(\theta) - g(\theta)\right] \tag{21}$$

$$V_\theta' + V_r = -K\left[f(\theta) + g'(\theta)\right] \tag{22}$$

In Ref. 8, the solution of this system is expressed as

$$-V_r = R\cos(\theta - \theta_0 + \alpha) + Kf(\theta) \tag{23}$$

$$V_\theta = R\sin(\theta - \theta_0 + \alpha) - Kg(\theta) \tag{24}$$

where

$$R = \sqrt{\left[V_{r_0} + Kf(\theta_0)\right]^2 + \left[V_{\theta_0} + Kg(\theta_0)\right]^2}$$

$$\tan\alpha = -\left[Kg(\theta_0) + V_{\theta_0}\right] / \left[Kf(\theta_0) + V_{r_0}\right]$$

In the present Note it is shown that the closed-form solution is possible when $F(\theta)$ and $G(\theta)$ are taken as any general function of θ. The solution of Yang and Yeh[8] can be easily obtained from this general solution by taking $F(\theta)$ and $G(\theta)$ as given by Eqs. (19) and (20).

IV. Analytical Solution of System (19) and (20) when $F(\theta)$ and $G(\theta)$ are Held as a General Function of θ

After changing the independent variable t to θ, we get

$$V_r' - V_\theta = F(\theta) \tag{25}$$

$$V_\theta' + V_r = G(\theta) \tag{26}$$

This is a simple first-order system in V_r and V_θ. By elimination of V_θ, we get

$$V_r'' + V_r = F'(\theta) + G(\theta) \tag{27}$$

The general solution of Eq. (27) is

$$V_r = A\cos\theta + B\sin\theta - \cos\theta \int_{\theta_0}^{\theta_1} \left[\frac{F'(\theta) + G(\theta)}{W}\right] \sin\theta \, d\theta$$

$$+ \sin\theta \int_{\theta_0}^{\theta_1} \left[\frac{F'(\theta) + G(\theta)}{W}\right] \cos\theta \, d\theta \tag{28}$$

where W is the Wornskian of the two independent solutions of the homogeneous equation (27). In our case, this is unity.

And, similarly,

$$V_\theta = \lambda\cos\theta + \lambda_2\sin\theta - \cos\theta \int_{\theta_0}^{\theta_1} \left[\frac{G'(\theta) - F(\theta)}{W}\right] \sin\theta \, d\theta$$

$$+ \sin\theta \int_{\theta_0}^{\theta_1} \left[\frac{G'(\theta) - F(\theta)}{W}\right] \cos\theta \, d\theta \tag{29}$$

where the constants A, B, and λ_1, λ_2 can be evaluated with the aid of the initial condition.

The relation of V_r and V_θ as given by Eqs. (28) and (29) is the solution of the system (6a) and (6b) when $F(\theta)$ and $G(\theta)$ are any generalized functions of θ.

The solution derived by Yang and Yeh can be readily obtained from relations (28) and (29) when $F(\theta)$ and $G(\theta)$ are expressed as given by relations (19) and (20). The same is shown below:

For the preceding case, the solutions for V_r and V_θ reduce to

$$V_r = A\cos\theta + B\sin\theta - Kf$$

$$V_\theta = \lambda_1\cos\theta + \lambda_2\sin\theta - gK$$

which can be expressed as

$$V_r = -R\cos(\theta + \mu) - Kf \tag{28a}$$

$$V_\theta = R\sin(\theta + \mu) - Kg \tag{29a}$$

where $\mu = (\alpha - \theta_0)$, which are the same as obtained by Yang and Yeh [Eqs. (8a) and (8b) of Ref. 8].

The other particular cases as derived by them would follow easily.

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References

[1]Murtaugh, S. A., and Criel, H. E., "Fundamentals of Proportional Navigation," *IEEE Spectrum,* Vol. 3, Dec. 1966, pp. 75–85.

[2]Guelman, M., "The Closed Form Solution of True Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. 12, July 1976, pp. 472–482.

[3]Shukla, U. S., and Mahapatra, P. R., "Generalised Linear Solution of Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. 24, No. 3, May 1988, pp. 231–238.

[4]Lin, C. F., and Tsai, L. L., "Analytical Solution of Optimal Trajectory Shaping Guidance," *Journal of Guidance, Control, and Dynamics,* Vol. 10, No. 1, 1987, pp. 61–66.

[5]Rao, M. N., "Analytical Solution of Optimal Trajectory—Shaping Guidance," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 4, 1989, pp. 600–601.

[6]Hough, M. E., "Explicit Guidance Along an Optimal Space Curve," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 4, 1989, pp. 495–504.

[7]Shukla, U. S., and Mahapatra, P. R., "The Proportional Navigation Dilemma—Pure or True," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. 26, No. 2, March 1990, pp. 382–392.

[8]Yang, C. D., Yeh, F. B., and Chen, J. H., "The Closed Form Solution of Generalized Proportional Navigation," *Journal of Guidance, Control, and Dynamics,* Vol. 10, No. 2, 1987, pp. 216–218.